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Relation between Nusselt number and Rayleigh number in turbulent thermal convection

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A theory is developed for the dependence of the Nusselt number on the Rayleigh number in turbulent thermal convection in horizontal fluid layers. The theory is based on a number of assumptions regarding the behaviour in the molecular boundary layers and on the assumption of a buoyancy-defect law in the interior analogous to the velocity-defect law in flow in pipes and channels. The theory involves an unknown constant exponent s and two unknown functions of the Prandtl number. For either $s = \frac{1}{2}$ or $s = \frac{1}{3}$, corresponding to two different theories of thermal convection, and for a given Prandtl number, constants can be chosen to give excellent agreement with existing data over nearly the whole explored range of Rayleigh numbers in the turbulent case. Unfortunately, comparisons with experiment do not permit a definite choice of s, but consistency with the chosen form of the buoyancy-defect law seems to suggest $s = \frac{1}{3}$, corresponding to similarity theory.

1. Derivation of Nusselt number–Rayleigh number equation

Recent experiments (Chu & Goldstein 1973; Garon & Goldstein 1973; Threlfall 1975) have yielded excellent data on the relationship between the Nusselt number Nu and the Rayleigh number Ra in thermal convection within horizontal fluid layers heated from below. These quantities are defined by

$$Nu = q \Big/ \frac{\kappa \Delta b}{H}, \quad Ra = \frac{\Delta b H^3}{\nu \kappa}, \tag{1}$$

where q is the buoyancy flux, κ is the coefficient of thermal conduction, ν is the viscosity, H is the fluid depth and Δb is the buoyancy difference between the plates, where buoyancy is defined by

$$b = \frac{(\rho - \rho_0)}{\rho_0} g. \tag{2}$$

In (2), ρ is density, ρ_0 is the density of the fluid at the lower plate and g is the acceleration due to gravity. With the Boussinesq approximation, dimensional considerations require that Nu be a function of Ra and the Prandtl number $Pr = \nu/\kappa$.

The experimental results have been somewhat disappointing to theorists. A plausible argument (Howard 1966) suggests that the buoyancy flux q should become independent of the depth as H gets large (large Ra), so that Nu should be proportional to $Ra^{\frac{1}{3}}$ at large Ra. Instead, the data indicate a relationship

$$Nu = \text{constant} \times Ra^n, \tag{3}$$

where n is unmistakably less than 0·333. Estimates of n range from 0·278 to 0·293. We shall show below that these observations are nevertheless consistent with the assumption that $Nu \propto Ra^{\frac{1}{3}}$ as $R \to \infty$.

In the development below, we match expressions for $\bar{b}$ in the interior of the fluid with the expressions for $\bar{b}$ in the molecular layers in the manner of Izakson (1937) and Millikan (1937) in their studies of turbulent shearing flow. If the Prandtl number is small, there will be a thin viscous layer imbedded in a thick conductive layer. If Pr is large, the viscous layer will be thicker. When the conductive layer is as thick as or thicker than the viscous layer, we may assume that advection and conduction of buoyancy are of the same order, i.e.

$$w_c' \delta_c / \kappa \sim 1, \tag{4}$$

where δ_c is the thickness of the conductive layer and w_c' is the disturbance vertical velocity in the layer. In addition, we assume that the buoyancy force and the inertial force are of the same order in the vertical equation of motion, so that

$$b_c' \sim w_c'^2 / \delta_c, \tag{5}$$

where b_c' is the buoyancy fluctuation. In view of our definition of ρ_0 in (2), $\bar{b} = 0$ at the lower plate and $\bar{b} = \Delta b$ at the upper plate. If we assume $\bar{b}_c \sim b_c'$, (4), (5) and $q \sim \kappa \bar{b}_c / \delta_c$ lead to

$$\delta_c \sim \nu^{\frac{3}{4}} q^{-\frac{1}{4}} Pr^{-\frac{3}{4}}, \tag{6}$$

$$\bar{b}_c = q^{\frac{3}{4}} \nu^{-\frac{1}{4}} Pr^{\frac{1}{4}} f_c(\xi_c, Pr), \tag{7}$$

$$\xi_c = z q^{\frac{1}{4}} \nu^{-\frac{3}{4}} Pr^{\frac{3}{4}}, \tag{8}$$

where $f_c(\xi_c, Pr)$ is independent of Pr as $Pr \to 0$.

When the viscous layer is as thick as or thicker than the conductive layer, we may assume a balance between viscous and inertial forces, i.e.

$$w_v' \delta_v / \nu \sim 1. \tag{9}$$

We may also assume $w_v' b_v' \sim q$ and $b_v' \sim \Delta \bar{b}_v$, where $\Delta \bar{b}_v$ is the increment in buoyancy across the viscous layer if $Pr \sim 1$ or the portion of the layer excluding the thermal boundary layer if $Pr \gg 1$. We assume, finally, that the buoyancy flux (production term) and dissipation term in the energy equation are of the same order, i.e.

$$q \sim \nu (w_v' / \delta_v)^2. \tag{10}$$

These lead to

$$\delta_v \sim \nu^{\frac{3}{4}} q^{-\frac{1}{4}}, \tag{11}$$

$$b_v' \sim \Delta \bar{b}_v \sim q^{\frac{3}{4}} \nu^{-\frac{1}{4}}, \tag{12}$$

$$\xi = z q^{\frac{1}{4}} \nu^{-\frac{3}{4}}. \tag{13}$$

Let us now investigate the imbedded thermal boundary layer. Here we assume a balance between advection and conduction again, or

$$w_c' \delta_c / \kappa \sim 1,$$

but we now assume that buoyancy forces and viscous forces are of the same order, or

$$b'_c \sim \nu w'_c/\delta_c^2.$$

Using $q \sim \kappa(\Delta\bar{b})_c/\delta_c$ and $b'_c \sim \Delta\bar{b}_c$, where $\Delta\bar{b}_c$ is the mean buoyancy difference across the imbedded thermal boundary layer, we get

$$\Delta\bar{b}_c \sim q^{\frac{3}{4}}\nu^{-\frac{1}{4}}Pr^{\frac{1}{2}}.$$

Since $\Delta\bar{b}_v$ is much less than the value of $\bar{b}$ at the top of the thermal boundary layer when Pr is large, we have

$$\bar{b}_v = C(Pr)\, q^{\frac{3}{4}}\nu^{-\frac{1}{4}}Pr^{\frac{1}{2}} + q^{\frac{3}{4}}\nu^{-\frac{1}{4}}f_v(\xi, Pr), \tag{14}$$

where $f_v(\xi, Pr)$ is independent of Pr as $Pr \to \infty$ and $C(Pr)$ is a constant at large values of Pr.

In the interior, we introduce a 'buoyancy-defect' law for $\bar{b}$ analogous to the velocity-defect law for flow in the interior of a pipe or channel (Monin & Yaglom 1971). We assume

$$\frac{\bar{b} - \frac{1}{2}\Delta b}{q/U} = -\chi\left(\frac{z}{H}, Pr\right), \tag{15}$$

where U is some unspecified velocity scale and where we allow variation with Pr. The basic assumption in (15) is the neglect of a non-dimensional number

$$(q^{\frac{1}{4}}\nu^{\frac{1}{4}})\, H/\nu$$

in the function χ, analogous to the neglect of the Reynolds number in the velocity-defect law in a pipe or channel. We may also define

$$m(\eta, Pr) = \frac{\frac{1}{2}\Delta b}{q^{\frac{3}{4}}\nu^{-\frac{1}{4}}}, \quad \eta = \frac{\nu^{\frac{3}{4}}}{Hq^{\frac{1}{4}}}, \quad \frac{U}{q^{\frac{1}{4}}\nu^{\frac{1}{4}}} = \alpha(\eta, Pr). \tag{16a–c}$$

Let us now match the expressions for $\bar{b}$ in the interior and in the layer near the plate, in which

$$\bar{b} = Cq^{\frac{3}{4}}\nu^{-\frac{1}{4}}\Delta + q^{\frac{3}{4}}\nu^{-\frac{1}{4}}\delta f(\xi\epsilon, Pr), \tag{17}$$

where

$$\Delta \to 0, \quad \delta \to Pr^{\frac{1}{4}}, \quad \epsilon \to Pr^{\frac{3}{4}} \quad \text{as} \quad Pr \to 0,$$
$$\Delta \to Pr^{\frac{1}{2}}, \quad \delta \to 1, \quad \epsilon \to 1 \quad \text{as} \quad Pr \to \infty$$

and $f(\xi\epsilon, Pr)$ is independent of Pr at these extremes. We obtain for the region of overlap

$$\alpha(\eta, Pr)\,[\delta f(\xi\epsilon, Pr) + C\Delta - m(\eta, Pr)] = -\chi(\eta\xi, Pr).$$

We now differentiate with respect to ξ and with respect to η and use the resulting two equations to eliminate the derivative of χ. We get

$$\alpha\delta\epsilon f'\xi = \eta\alpha_\eta(\delta f + C\Delta) - \eta(\alpha m)_\eta, \tag{18}$$

where the prime denotes partial differentiation with respect to $\xi\epsilon$ and the subscripts also indicate partial derivatives. If we differentiate (18) with respect to ξ, we get

$$(f'\epsilon\xi)'/f' = \eta\alpha_\eta/\alpha = -s, \tag{19}$$

where s is a constant. Solutions are

$$f = \tfrac{1}{2}A_0 + B_0(\xi\epsilon)^{-s}, \quad \alpha = \beta\eta^{-s}, \tag{20}$$

where A_0, B_0 and β are functions of Pr. The factor $\frac{1}{2}$ is introduced for later convenience. Equation (18) now yields

$$m = \tfrac{1}{2}A_0\delta + C\Delta - \tfrac{1}{2}\gamma_0\eta^s, \tag{21}$$

where γ_0 is another function of Pr. We may now use (16*a*) to obtain the Nusselt number:

$$Nu = \frac{Ra^{\frac{1}{3}}Pr^{\frac{2}{3}}}{[A_0\delta + 2C\Delta - \gamma(NuRa)^{-\frac{1}{4}s}]^{\frac{4}{3}}}, \tag{22}$$

where γ is a function of Pr. The physical argument that the velocity scale should not increase with an increase in viscosity indicates that $s \geqslant \frac{1}{3}$, so that the buoyancy flux becomes independent of H for large Rayleigh numbers. When Pr is large or small, A_0 and C are independent of Pr. If, in addition, Ra is large, we get

$$\left.\begin{aligned} Nu &= K_1Ra^{\frac{1}{3}}, \quad Ra \text{ large} \quad Pr \text{ large},\\ Nu &= K_2Ra^{\frac{1}{3}}Pr^{\frac{1}{3}}, \quad Ra \text{ large}, \quad Pr \text{ small},\end{aligned}\right\} \tag{23}$$

where K_1 and K_2 are constants. This agrees with the theory of Kraichnan (1962).

2. Comparison with experiment

We may try to determine s, $A_0\delta + 2C\Delta$ and γ from experiments. The best data, perhaps, are experimental measurements of heat flux in gaseous helium by Threlfall (1975) and in water by Garon & Goldstein (1973). We make two choices for s, namely $s = \frac{1}{2}$ and $s = \frac{1}{3}$, corresponding respectively to a recent theory of the author (Long 1975) and to the similarity theory of Prandtl (1932) and Priestley (1954). For helium ($Pr = 0{\cdot}8$), we obtain

$$Nu = \begin{cases} \left.\dfrac{0{\cdot}0524Ra^{\frac{1}{3}}}{[1 - 1{\cdot}021(RaNu)^{-\frac{1}{12}}]^{\frac{4}{3}}}\right\} & \text{for} \quad s = \frac{1}{3}, \\[2ex] \left.\dfrac{0{\cdot}0569Ra^{\frac{1}{3}}}{[1 - 1{\cdot}64(RaNu)^{-\frac{1}{8}}]^{\frac{4}{3}}}\right\} & \text{for} \quad s = \frac{1}{2}, \end{cases} \tag{24}$$

where we have used two fairly extreme experiments of Threlfall, with

$$\left.\begin{aligned} Nu &= 8, \quad Ra = 10^6,\\ Nu &= 63, \quad Ra = 10^9.\end{aligned}\right\} \tag{25}$$

Threlfall's experiments included measurements at much lower Rayleigh numbers, but transitions, or discontinuities, in the curve $Nu = f(Ra)$ are observed below $Ra = 10^6$, and these are not revealed in the present theory. Garon & Goldstein believe that their data reveal abrupt changes in the *slope* of the curve at Rayleigh numbers above 10^6 but this is less certain.

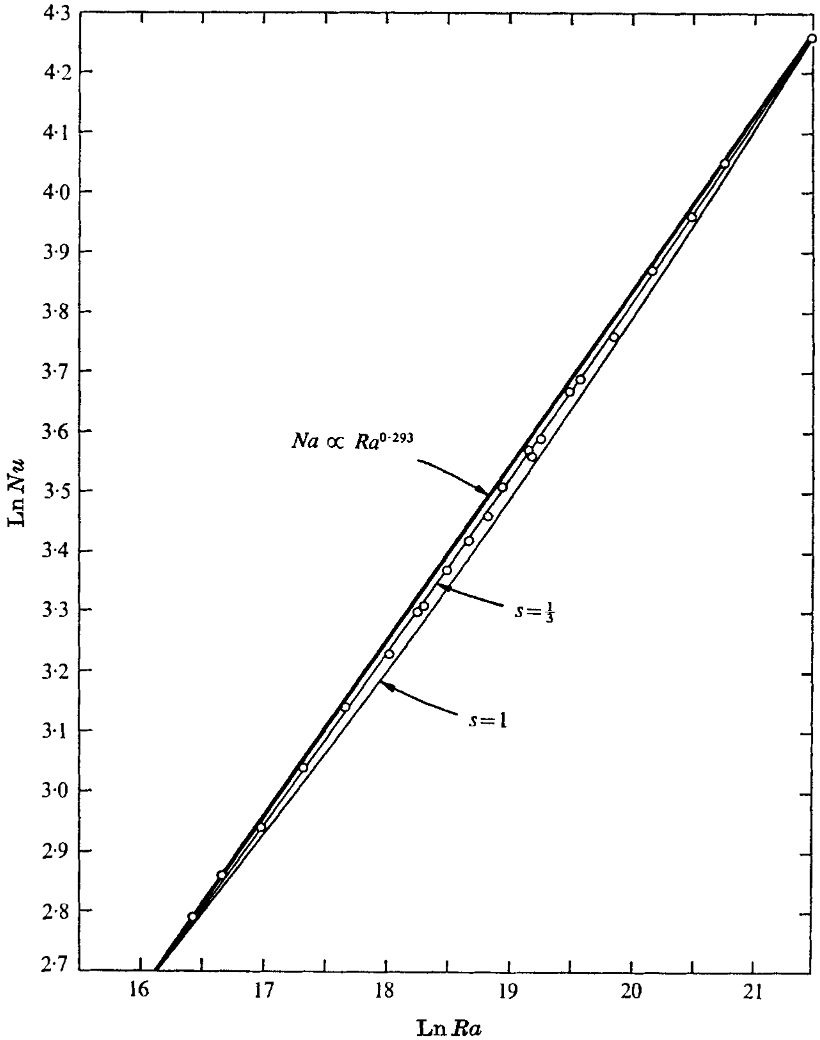

FIGURE 1. Comparison of the data of Garon & Goldstein, the theoretical curves for $s = \frac{1}{3}$ and $s = 1$, and the power law (straight line).

An additional comparison may be made using the data of Garon & Goldstein for water ($Pr = 6{\cdot}8$). We get

$$Nu = \begin{cases} \dfrac{0{\cdot}04356 Ra^{\frac{1}{3}}}{[1-1{\cdot}402(RaNu)^{-\frac{1}{18}}]^{\frac{4}{3}}} & \text{for} \quad s = \frac{1}{3}, \\ \left.\dfrac{0{\cdot}04786\, Ra^{\frac{1}{3}}}{[1-2{\cdot}544(RaNu)^{-\frac{1}{8}}]^{\frac{4}{3}}}\right\} & \text{for} \quad s = \frac{1}{2}, \end{cases} \tag{26}$$

where we have used the lowest and highest Rayleigh numbers:

$$\left.\begin{aligned} Nu &= 16{\cdot}2, \quad Ra = 1{\cdot}36\times 10^{7}, \\ Nu &= 81{\cdot}3, \quad Ra = 3{\cdot}29\times 10^{9}. \end{aligned}\right\} \tag{27}$$

In figure 1 we have drawn theoretical curves for $s = \frac{1}{3}$ and $s = 1$, together with the data of Garon & Goldstein and a straight line representing a power law through the two observations in (27). The straight line is in fairly good agreement with the data and this caused Garon & Goldstein to choose a power law $Nu \propto Ra^{0\cdot293}$. The theory for $s = \frac{1}{3}$ gives better agreement than either the power law (which has no theoretical basis) or the theory for $s = 1$ (corresponding to a theory of Malkus 1954). We do not show the curve for $s = \frac{1}{2}$ because it is identical to the curve for $s = \frac{1}{3}$ when plotted on this scale. Threlfall presented a plot of his heat-flux measurements and there is good agreement with the present theory and the power law $Nu \propto Ra^{0\cdot280}$. Threlfall did not supply numerical data and a close comparison with the present theory is not possible.

We have remarked that the theory of this paper indicates $Nu \propto Ra^{\frac{1}{3}}$ as $Ra \rightarrow \infty$, but there are sensible departures at the Rayleigh numbers of the typical experiment. We can see how large the Rayleigh number must be for (26) to yield a close approximation to the $Ra^{\frac{1}{3}}$ law. For $s = \frac{1}{3}$, we get $Ra > 3 \times 10^{21}$ for an error of less than 1 % in the Nusselt number. The corresponding value for $s = \frac{1}{2}$ is

$$Ra > 1\cdot4 \times 10^{16}.$$

Equations (24) give even larger Rayleigh numbers.

3. Discussion and summary

In a recent paper (Long 1975), I have advanced a theory of convection as an alternative to the similarity theory of Prandtl (1932) and Priestley (1954). In the older theory, it is assumed that all variables become independent of molecular quantities in a region $\delta_m \ll z \ll H$, where δ_m is the thickness of the thicker of the two boundary layers, and therefore, all variables depend only on q and z. This implies that scales in the interior depend only on q and H and this requires that $s = \frac{1}{3}$ in (20). My theory was motivated by certain observations in the atmospheric surface layer and in laboratory experiments which seem to differ from the similarity theory. With regard to the present investigation, it certainly appears likely that the Prandtl number, which is the ratio of the molecular coefficients, is an important quantity even at large Rayleigh numbers, but a variation of the similarity theory in which all variables in the interior depend on q, H and Pr only certainly is more in keeping with the buoyancy-defect law adopted in (15). Indeed, it seems inconsistent to neglect the quantity η in the function χ in (15) but allow U, and therefore the left-hand side of (15), to involve molecular quantities directly. It is disappointing that observations do not permit a clear-cut choice of s, but I am at present somewhat reluctantly inclined to the conclusion that the similarity theory may be correct after all, at least at very high Rayleigh numbers.

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